CAUSAL PROPAGATION OF CONSTRAINTS AND THE CANONICAL FORM OF GENERAL RELATIVITY

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Abstract

Studies of new hyperbolic systems for the Einstein evolution equations show that the "slicing density" $\alpha(t,x)$ can be freely specified while the lapse $N = \alpha g^{1/2}$ cannot. Implementation of this small change in the Arnowitt-Deser-Misner action principle leads to canonical equations that agree with the Einstein equations whether or not the constraints are satisfied. The constraint functions, independently of their values, then propagate according to a first order symmetric hyperbolic system whose characteristic cone is the light cone. This result follows from the twice-contracted Bianchi identity and constitutes the central content of the constraint "algebra" in the canonical formalism.

In this paper* I will show one way to follow a path to some small, but striking improvements in the Arnowitt-Deser-Misner (ADM) canonical form of general relativity. [1, 2] Arlen Anderson, collaborator in the primary reference [3] on this subject, and I followed a different track in the presentation given elsewhere. [3]

Several aspects of the ADM canonical form of general relativity [1, 2] have been improved recently. [3] Slight modifications give rise to possibly significant changes of perspective. The modifications were suggested by the structure of the evolution equations written in new, manifestly hyperbolic, forms, not directly relevant to the present work, in which only physical characteristic speeds appear. [4, 5, 6, 7, 8, 9, 10, 11] One may see some aspects of those results as flowing to, or from, the action principle.

Here I focus on the consequences of a direct demonstration that the constraint functions, whether they vanish or not, always propagate as dictated by a first order symmetric hyperbolic system whose only characteristic cone is the

*Dedicated to Richard Arnowitt

full physical light cone. This system is in fact just a new reading of the twice-contracted Bianchi identities. This reading suggests the form that the canonical momentum evolution equations ought to take. This observation, in turn, causes one to change the treatment of the lapse function N; and this change leads to a small alteration in the action principle. One's view of general relativity is then changed by noting that parameter time differentiation of the canonical variables now holds correctly throughout the entire phase space, among other things.

The definition of the Einstein tensor $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$ in terms of the Ricci tensor $R_{\mu\nu}$ leads to the observation that $2G_0^0 \equiv R_0^0 - R_k^k$ and to the important identity

$$G_{ij} + g_{ij}G_0^0 \equiv R_{ij} - g_{ij}R_k^k \ . \tag{1}$$

The vanishing of the right hand side does not depend on any constraints and is equivalent to $R_{ij} = 0$. Evidently, $R_{ij} = 0$ is also equivalent to $G_{ij} = -g_{ij}G_0^0$. Therefore, while $R_{\mu\nu} = 0$ and $G_{\mu\nu} = 0$ are equivalent, $R_{ij} = 0$ and $G_{ij} = 0$ are not equivalent equations of motion unless the Hamiltonian constraint

$$\mathcal{H} \equiv g^{1/2}\mathcal{C} \equiv 2g^{1/2}G_0^0 \equiv g^{1/2}(K_{ij}K^{ij} - H^2 - \bar{R}) = 0 \tag{2}$$

holds exactly. (Here, $H = K_j^j$ denotes the trace of the extrinsic curvature K_{ij} of a spacelike "time slice" with metric g_{ij}, density $g^{1/2} = (\det g_{ij})^{1/2}$, scalar curvature $\bar{R}$, and spatial covariant derivative $\bar{\nabla}_i$.)

These observations lead immediately to a transparent form of the twice-contracted Bianchi identities $\nabla_\beta G_\alpha^\beta \equiv 0$. Namely,

$$\nabla_\beta G_0^\beta \equiv \nabla_0 G_0^0 + \nabla_j G_0^j \equiv 0 \ , \tag{3}$$

$$\nabla_\beta G_j^\beta \equiv \nabla_0 G_j^0 - \nabla_j G_0^0 + \nabla_i \left[R_j^i - \delta_j^i R_k^k\right] \equiv 0 \ . \tag{4}$$

Recall that the momentum constraints are given by

$$\mathcal{H}_i \equiv g^{1/2}\mathcal{C}_i \equiv 2g^{1/2}NR_i^0 \equiv 2g^{1/2}NG_i^0 = 0 \ . \tag{5}$$

Let us write the Bianchi identities (3), (4) in $3+1$ form using the spacetime cobasis indicated by parentheses in the spacetime metric

$$ds^2 = -N^2\,(dt)^2 + g_{ij}\left(dx^i + \beta^i dt\right)\left(dx^j + \beta^j dt\right) \ , \tag{6}$$

where N is the lapse function and β^i is the shift vector. The dual vector basis is $\partial_0 = \partial_t - \beta^j\partial_j$, with $\partial_j \equiv \partial/\partial x^j$ and $\partial_t = \partial/\partial t$. We define our time derivative operator on time-dependent spatial tensors by $\hat{\partial}_0 = \text{(over-dot)} = \partial_t - \pounds_{\boldsymbol{\beta}}$, with $\pounds_{\boldsymbol{\beta}}$ the spatial Lie derivative along the shift vector. We also use the spatial covariant derivative $\bar{\nabla}_j$.

From the cobasis $\left\{\theta^0 = dt,\ \theta^i = dx^i + \beta^i dt\right\}$ and $d\theta^\alpha = -\frac{1}{2}C^\alpha{}_{\beta\gamma}\theta^\beta \bigwedge \theta^\gamma$ we find $C^i{}_{0j} = -C^i{}_{j0} = \partial_j\beta^i$ and all other structure coefficients vanish. This gives

the connection coefficients

$$
\begin{aligned}
\gamma^{i}{}_{jk} &= \Gamma^{i}{}_{jk} = \bar{\Gamma}^{i}_{jk} \\
\gamma^{i}{}_{0j} &= -NK^{i}{}_{j} \\
\gamma^{i}{}_{j0} &= -NK^{i}{}_{j} + \partial_j \beta^i \\
\gamma^{i}{}_{00} &= N\partial^i N \\
\gamma^{0}{}_{ij} &= -N^{-1}K_{ij} \\
\gamma^{0}{}_{0i} &= \gamma^{0}{}_{i0} = \partial_i \log N \\
\gamma^{0}{}_{00} &= \partial_0 \log N \ ,
\end{aligned} \tag{7}
$$

where Γ denotes a Christoffel symbol. Our derivative convention is $\nabla_\alpha \sigma_\beta = \partial_\alpha \sigma_\beta - \sigma_\rho \gamma^{\rho}{}_{\beta\alpha}$.

Now we can go through a straightforward process of writing (3) and (4) in terms of $\hat{\partial}_0\,(\ ) = \dot{(\ )}$ and $\bar{\nabla}_j$. For the unweighted constraint functions $\mathcal{C}$ and $\mathcal{C}_i$, the Bianchi identities then become

$$
\dot{\mathcal{C}} - N\bar{\nabla}^j \mathcal{C}_j \equiv 2\left(\mathcal{C}_j \bar{\nabla}^j N + NH\mathcal{C} - NK_{ij}\left[\mathcal{R}_{ij}\right]\right) \ , \tag{8}
$$

$$
\dot{\mathcal{C}}_j - N\bar{\nabla}_j \mathcal{C} \equiv 2\left(\mathcal{C}\bar{\nabla}_j N + \frac{1}{2}NH\mathcal{C}_j - \bar{\nabla}^i\left(N\left[\mathcal{R}_{ij}\right]\right)\right) \ , \tag{9}
$$

where $\mathcal{R}_{ij} \equiv R_{ij} - g_{ij}R^k_k$. Similar results hold for $\mathcal{H} = g^{1/2}\mathcal{C}$ and $\mathcal{H}_i = g^{1/2}\mathcal{C}_i$. (Related formulas were found by Choquet-Bruhat and Noutchegueme [12] in studying the evolution of matter sources ρ^{00}, ρ^{0i}, where $\rho^{\beta\alpha} = T^{\beta\alpha} - \frac{1}{2}g^{\beta\alpha}T^{\mu}_{\mu}$.)

The system (8), (9) is first order symmetric ("symmetrizable") hyperbolic with characteristic fields consisting of linear combinations of $\mathcal{C}$ and $\mathcal{C}_i$ propagating along the light cone. Hence, if the equations of motion $\mathcal{R}_{ij} = 0$ (or $R_{ij} = 0$) hold, $\mathcal{C}$ and $\mathcal{C}_i$ remain zero in the physical domain of dependence associated with the region of the initial data surface on which they were initially satisfied. If none of $\mathcal{C}$, $\mathcal{C}_i$, $\mathcal{R}_{ij}$ is exactly zero, as is inevitably the case in approximations and numerical applications, then (8), (9) show how errors in $\mathcal{C}$ and $\mathcal{C}_i$ are driven along the light cone by terms linear in these errors and by $R_{ij} - g_{ij}R^k_k$. (It is straightforward to add a cosmological constant and a matter source $T^{\alpha\beta}$, such that $\nabla_\beta T^{\beta\alpha} = 0$, to this analysis.)

To draw some lessons for the canonical formalism, let us first express the $3+1$ evolution equations in their standard geometrical form (with zero shift, Ref. 13; arbitrary lapse and shift, Ref. 14; spacetime perspective, Ref. 15):

$$
\dot{g}_{ij} \equiv -2NK_{ij} \ , \tag{10}
$$

$$
\dot{K}_{ij} \equiv N\left(-R_{ij} + \bar{R}_{ij} + HK_{ij} - K_{ik}K^k_j - N^{-1}\bar{\nabla}_i \partial_j N\right) \ . \tag{11}
$$

A brief look at (11) shows that the term $\mathcal{R}_{ij} = \left(R_{ij} - g_{ij}R^k_k\right)$ that appears in the hyperbolic form (8), (9) of the Bianchi identities is made-to-order to

produce an equation of motion for the ADM canonical momentum

$$\pi^{ij} = g^{1/2}\left(Hg^{ij} - K^{ij}\right) \tag{12}$$

that *contains no constraints.* Indeed, using (10) and (11), we obtain the *identity*

$$\begin{aligned}\dot{\pi}^{ij} \quad \equiv \quad & Ng^{1/2}\left(\bar{R}g^{ij} - \bar{R}^{ij}\right) - Ng^{-1/2}\left(2\pi^{ik}\pi_k^j - \pi\pi^{ij}\right) \\ & + g^{1/2}\left(\bar{\nabla}^i\bar{\nabla}^j N - g^{ij}\bar{\nabla}_k\bar{\nabla}^k N\right) + Ng^{1/2}\left[\mathcal{R}^{ij}\right] \ . \end{aligned} \tag{13}$$

From (6), we have

$$\dot{g}_{ij} \equiv Ng^{-1/2}\left(2\pi_{ij} - \pi g_{ij}\right) \ . \tag{14}$$

If we now compute the time derivatives of $\mathcal{C}(\mathbf{g}, \boldsymbol{\pi})$ and $\mathcal{C}_i(\mathbf{g}, \boldsymbol{\pi})$ using (13), (14), we obtain, of course, the Bianchi identities in the form (8), (9). This merely stresses that the Bianchi identities, properly construed, are just the hyperbolic evolution equations for those phase space functions whose vanishing yields the "constraint hypersurface" in phase space, and in particular we see that (8), (9) hold on or off the constraint hypersurface.

We have now reached a crucial point in the development. If the canonical equation for $\dot{\pi}^{ij}$ is dictated by the vanishing of the spatial part of the Einstein tensor, $G^{ij} = 0$, as in the ADM analysis, [1] then the identities (1) and (13) show that the constraint term $(1/2)\, g^{ij} g^{1/2}\mathcal{C} \equiv (1/2)\, g^{ij}\mathcal{H}$ remains in the $\dot{\pi}^{ij}$ equation, restricting its validity to the subspace of phase space on which the constraints are satisfied (*i.e.*, where the constraint functions vanish). Furthermore, if we substitute $G^{ij} = 0$ back into the Bianchi identities (3) and (4), then the hyperbolicity and well-posedness of the constraint evolution are lost. (The latter has been observed by Frittelli using different methods. [16] She also found that $R_{ij} = 0$ gives well-posed evolution.)

However, though the ADM derivation of the $\dot{\pi}^{ij}$ equation, found by varying g_{ij} in their canonical action $(16\pi G = c = 1)$

$$S\left[\mathbf{g}, \boldsymbol{\pi}; N, \boldsymbol{\beta}, \right) = \int d^4x\left(\pi^{ij}\dot{g}_{ij} - N\mathcal{H}\right) \ , \tag{15}$$

with $N(t,x)$ and $\beta^i(t,x)$ as undetermined multipliers, is of course perfectly correct, another point of view is possible. (We are ignoring boundary terms, which are not of interest here, and we note that the momentum constraint term $-\beta^i\mathcal{H}_i$ $(\mathcal{H}_i = g^{1/2}\mathcal{C}_i)$ is contained in $\pi^{ij}\dot{g}_{ij}$ $(\dot{(\ )} \equiv \hat{\partial}_0)$ upon integration by parts.) The other point of view arises in our work on hyperbolic systems with only *physical* characteristics. [4, 5, 6, 7, 8, 9, 10, 11] There, one is *never* allowed to specify $N(t,x)$ freely! Rather one must use in essence Choquet-Bruhat's "algebraic gauge," which asserts that the weight-minus-one lapse (the "slicing density") $\alpha = Ng^{-1/2} = \alpha(t,x) > 0$ is *freely* specifiable while N is not. [17, 4] (The slicing density α has been used prominently in the action by Teitelboim [18] and Ashtekar [19, 20] for other purposes.)

Indeed, if one computes $\hat{\partial}_0 \log\alpha = f(t,x)$ from a given $\alpha(t,x)$, then

$$g^{1/2}\hat{\partial}_0\alpha = \hat{\partial}_0 N + N^2 H = Nf \ , \tag{16}$$

which is the equation of harmonic time slicing with $f(t,x) = \hat{\partial}_0 \log\alpha$ acting as a "gauge source." [21] In this sense the undetermined multiplier $\alpha(t,x)$, like $\beta^i(t,x)$, cannot affect the issue of hyperbolicity. [3, 4, 5, 6, 7, 8, 9, 10, 11] The "harmonic time slicing gauge" is not a gauge at all; it is simply the equation of motion of the lapse function (with a "gauge source") once one recognizes α as the true undetermined multiplier.

Combining (16) and the equation for $\dot{H}$ obtained from (11), one obtains a quasi-linear wave equation (hyperbolic with characteristic speed $c = 1$!) for N. Therefore, *every* foliation of a globally hyperbolic spacetime by regular time slices is given by some initial time slice, and the solution $N > 0$ of a wave equation, for some $\alpha(t,x) > 0$. *No* slices are "lost" by changing from N to α, so the latter is as "good" as the original N. (Degenerate cases can be handled as well with $g^{1/2}$ and α as with N.) We conclude that N is a *dynamical variable* (as did Ashtekar [19, 20] for other reasons) which determines the proper time $N\delta t$ between slices $t = t'$ and $t = t' + \delta t$. It is determined from $\alpha(t,x)$ and the solution $g^{1/2}$ of the constraint equations. [15, 22, 23, 24] From this perspective, the Hamiltonian constraint plays its familiar role as an initial-value constraint that determines $g^{1/2}$ given the other free data. [15] The important point is that $\mathcal{H} = 0$ does not determine the time but does fix the rate of proper time τ with respect to parameter time t: $d\tau/dt = g^{1/2}\alpha$ along the normal ∂_0.

Motivated by the above findings and those with respect to the $\dot{\pi}^{ij}$ equation and the well-posedness of the Bianchi equations as a causal hyperbolic system, we alter the undetermined multiplier N in the ADM action principle to α. Thus the Hamiltonian function becomes

$$\tilde{\mathcal{H}} = g^{1/2}\mathcal{H} = \pi_{ij}\pi^{ij} - \frac{1}{2}\pi^2 - g\bar{R} \ , \tag{17}$$

where $\tilde{\mathcal{H}}$ is a rational function of the metric of scalar weight plus two. The ADM-plus-Teitelboim-plus-Ashtekar action becomes

$$S\left[\mathbf{g}, \boldsymbol{\pi}; \alpha, \boldsymbol{\beta}\right) = \int d^4x \left(\pi^{ij}\dot{g}_{ij} - \alpha\tilde{\mathcal{H}}\right) \ . \tag{18}$$

The modified action *principle* for the canonical equations that we propose is to vary π^{ij} and g_{ij}, with $\alpha(t,x)$ and $\beta^i(t,x)$ as undetermined multipliers. From

$$\begin{aligned} \delta\tilde{\mathcal{H}} \quad = \quad & (2\pi_{ij} - g_{ij}\pi)\,\delta\pi^{ij} + \left(2\pi^{ik}\pi_k^j - \pi\pi^{ij} + g\bar{R}^{ij} - gg^{ij}\bar{R}\right)\delta g_{ij} \\ & - g\left(\bar{\nabla}^i\bar{\nabla}^j\delta g_{ij} - g^{ij}\bar{\nabla}_k\bar{\nabla}^k\delta g_{ij}\right) \end{aligned} \tag{19}$$

we obtain the canonical equations

$$\dot{g}_{ij} = \alpha\frac{\delta\tilde{\mathcal{H}}}{\delta\pi^{ij}} \quad = \quad \alpha\left(2\pi_{ij} - \pi g_{ij}\right) \equiv -2NK_{ij} \ , \tag{20}$$

$$\dot{\pi}^{ij} = -\alpha \frac{\delta \tilde{\mathcal{H}}}{\delta g_{ij}} = -\alpha g \left(\bar{R}^{ij} - \bar{R} g^{ij} \right) - \alpha \left(2\pi^{ik} \pi_k^j - \pi \pi^{ij} \right) + g \left(\bar{\nabla}^i \bar{\nabla}^j \alpha - g^{ij} \bar{\nabla}_k \bar{\nabla}^k \alpha \right) . \tag{21}$$

Equation (21) for $\dot{\pi}^{ij}$ is the identity (13) with $\mathcal{R}^{ij} = 0$, that is, $R^{ij} = 0$. Thus, (21) is a "strong" equation unlike its ADM counterpart which requires in addition the strict validity of a constraint: $\mathcal{H} = 0$.

In the present formulation, the canonical equations of motion hold everywhere on phase space with any parameter time t, a necessary condition for the issue of "constraint evolution" to be discussed properly at all in the Hamiltonian framework. Further, there seems to be no need to vary α and β^i in a spacetime action volume integral to enforce the constraints for *all* time, because the constraints are dynamically determined to hold in the appropriate physical domain of dependence if they hold initially, as shown below in (26), (27). Perhaps α and β^i should be varied on an initial slice only.

If we define the integrated or "smeared" Hamiltonian constraint as

$$\tilde{\mathcal{H}}_\alpha = \int d^3x' \alpha(t, x') \tilde{\mathcal{H}} , \tag{22}$$

the equation of motion for a general functional $F[\mathbf{g}, \boldsymbol{\pi}; t, x)$ anywhere on the phase space is

$$\dot{F}\left[\mathbf{g}, \boldsymbol{\pi}; t, x\right) = -\left\{ \tilde{\mathcal{H}}_\alpha, F \right\} + \tilde{\partial}_0 F , \tag{23}$$

where $\dot{( )}$ denotes our total time derivative and $\tilde{\partial}_0$ is a "partial" derivative of the form $\partial_t - \pounds_{\boldsymbol{\beta}}$ acting only on explicit spacetime dependence. The Poisson bracket is

$$\{F, G\} = \int d^3x \left(\frac{\delta F}{\delta g_{ij}(t,x)} \frac{\delta G}{\delta \pi^{ij}(t,x)} - \frac{\delta G}{\delta g_{ij}(t,x)} \frac{\delta F}{\delta \pi^{ij}(t,x)} \right) . \tag{24}$$

It is clear that the $(\dot{\mathbf{g}}, \dot{\boldsymbol{\pi}})$ equations come from (23) applied to the canonical variables. The harmonic time slicing equation (16) results from application of (23) to N, and the wave equation for N comes from a repeated application of (23) to (16).

Time evolution is generated by the Hamiltonian vector field

$$\mathcal{X}_{\tilde{\mathcal{H}}_\alpha} = \int d^3x \left\{ \alpha(2\pi_{ij} - \pi g_{ij}) \frac{\delta}{\delta g_{ij}} - [\alpha g(\bar{R}^{ij} - \bar{R} g^{ij}) + \alpha(2\pi^{ik}\pi_k^j - \pi\pi^{ij}) - g(\bar{\nabla}^i \bar{\nabla}^j \alpha - g^{ij} \bar{\nabla}_k \bar{\nabla}^k \alpha)] \frac{\delta}{\delta \pi^{ij}} \right\} . \tag{25}$$

Because is does not contain any explicit constraint dependence, (25) is a valid time evolution operator on the entire phase space.

The product rule applicable to $\hat{\partial}_0$ and the Poisson bracket shows that the evolution equations (8), (9) for the constraints can be written as

$$\hat{\partial}_0 \tilde{\mathcal{H}} = -\left\{\tilde{\mathcal{H}}_\alpha, \tilde{\mathcal{H}}\right\} = \alpha g g^{ij} \partial_i \mathcal{H}_j + 2 g g^{ij} \mathcal{H}_i \bar{\nabla}_j \alpha \ , \tag{26}$$

$$\hat{\partial}_0 \mathcal{H}_j = -\left\{\tilde{\mathcal{H}}_\alpha, \mathcal{H}_j\right\} = \alpha \partial_j \tilde{\mathcal{H}} + 2\tilde{\mathcal{H}} \partial_j \alpha \ , \tag{27}$$

where $\bar{\nabla}_j \alpha = \partial_j \alpha + \alpha g^{-1/2} \partial_i g^{1/2}$. Therefore, the Poisson brackets of the smeared with the unsmeared constraints are well-posed evolution equations for the constraints. These equations are, of course, just the twice-contracted Bianchi identities when $\mathcal{R}_{ij} = 0$ or $R_{ij} = 0$.

The consistency of time evolution with respect to different choices of α is checked by using the Jacobi identity,

$$\left\{\tilde{\mathcal{H}}_{\alpha_1}, \left\{\tilde{\mathcal{H}}_{\alpha_2}, F\right\}\right\} - \left\{\tilde{\mathcal{H}}_{\alpha_2}, \left\{\tilde{\mathcal{H}}_{\alpha_1}, F\right\}\right\} = \left\{\left\{\tilde{\mathcal{H}}_{\alpha_1}, \tilde{\mathcal{H}}_{\alpha_2}\right\}, F\right\} \ , \tag{28}$$

and

$$\left\{\tilde{\mathcal{H}}_{\alpha_1}, \tilde{\mathcal{H}}_{\alpha_2}\right\} = -\int d^3x g g^{ij} \left(\alpha_1 \partial_i \alpha_2 - \alpha_2 \partial_i \alpha_1\right) \mathcal{H}_j \ . \tag{29}$$

The metric dependence of (29) shows that the difference between evolution with α_2 followed by α_1, and *vice versa*, is a spatial diffeomorphism when $\mathcal{H}_i = 0$ or when $\delta F / \delta \pi^{ij} = 0$.

The results of the previous paragraphs shed new light on the Dirac "algebra" of constraints (cf. Ref 25). It is well known that the Dirac algebra is not the spacetime diffeomorphism algebra. This can be seen from the fact that while the action (18) is invariant under transformations generated by the constraint functions even when they do not vanish, [26] the equations of motion that follow from this action when the constraint functions do not vanish are $R_{ij} = 0$. These equations are preserved by spatial diffeomorphisms and time translations along their flow in phase space, but a general spacetime diffeomorphism applied to $R_{ij} = 0$ mixes in the constraints. A comparison of (8), (9) with (26), (27) shows this effect: the Bianchi identities are spacetime diffeomorphism covariant while (26), (27) are not. The equations (8), (9) and (26), (27) differ precisely by terms proportional to $\mathcal{R}_{ij}$ and $\bar{\nabla}^i \mathcal{R}_{ij}$.

A second important view of the Dirac algebra results from the direct and beautiful dynamical meaning of its once-smeared form. Equations (26) and (27) express consistency of the constraints as a well posed initial-value problem. If the constraint functions vanish initially, they continue to do so by evolution into the domain of dependence corresponding to the region of the initial time slice on which they initially vanished. This mechanism follows from the dual role of $\tilde{\mathcal{H}}$ as a constraint and as part of the generator of time translations of functionals of the canonical variables anywhere on the phase space.

Let us take note that the Hamiltonian constraint *per se* does not express the dynamics of the theory; the equation of dynamics is (23). In its "altered"

role, the Hamiltonian constraint function simply vanishes as an initial value condition, from which $g^{1/2}$ is determined as in the intial value problem. [15] Then N can be constructed from α. The Hamiltonian constraint, once solved, remains so according to the rigorous result embodied in (26), (27).

The application of these ideas to canonical quantum gravity will appear elsewhere. [27]

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References

[1] R. Arnowitt, S. Deser, and C.W. Misner, in *Gravitation: An Introduction to Current Research*, ed. L. Witten, (Wiley: New York, 1962), 227.

[2] P.A.M. Dirac, Proc. Roy. Soc. London **A246**, 333 (1958); Phys. Rev. **114**, 924 (1959).

[3] Arlen Anderson and James W. York, Jr., Phys. Rev. Letters, *to appear* (1998).

[4] Y. Choquet-Bruhat and J. W. York, C. R. Acad. Sci. Paris **t. 321**, Série I, 1089 (1995).

[5] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J.W. York, Phys. Rev. Lett. **75**, 3377 (1995).

[6] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J.W. York, C. R. Acad. Sci. Paris **t. 323**, Série II *b*, 835 (1996).

[7] Y. Choquet-Bruhat and J. W. York, Banach Center Publications **41**, Part 1, 119 (1997).

[8] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J.W. York, Class. Quantum Grav. **14**, A9 (1997).

[9] A. Abrahams, A. Anderson, Y. Choquet-Bruhat, and J.W. York, to appear in Proc. 1997 Texas Symposium on Relativistic Astrophysics, gr-qc/9703010 (1997).

[10] A. Anderson, Y. Choquet-Bruhat, and J.W. York, Topol. Meth. Nonlinear Anal. **10**, 353 (1997).

[11] Y. Choquet-Bruhat, J.W. York, and A. Anderson, to appear in Proc. 1997 Marcel Grossmann Meeting, gr-qc/9802027 (1998).

[12] Y. Choquet-Bruhat and N. Noutchegueme, C. R. Acad. Sci. Paris, **t. 303**, Série I, 259 (1986).

[13] A. Lichnerowicz, J. Math. Pures et Appl. **23**, 37 (1944).

[14] Y. Choquet (Foures)-Bruhat, J. Rat. Mechanics and Anal. **5**, 951 (1956).

[15] J.W. York, in *Sources of Gravitational Radiation*, ed. L. Smarr, (Cambridge Univ. Press: Cambridge, 1979).

[16] S. Frittelli, Phys. Rev. D **55**, 5992 (1997).

[17] Y. Choquet-Bruhat and T. Ruggeri, Commun. Math. Phys. **89**, 269 (1983).

[18] C. Teitelboim, Phys. Rev. D **25**, 3159, (1982); Phys. Rev. D **28**, 297 (1983).

[19] A. Ashtekar, *New Perspectives in Canonical Gravity*, (Bibliopolis: Naples, 1988).

[20] A. Ashtekar, Phys. Rev. D **36**, 1587 (1987).

[21] H. Friedrich, Commun. Math. Phys. **100**, 525 (1985).

[22] J.W. York, J. Math. Phys. **14**, 456 (1973).

[23] N. O'Murchadha and J.W. York, Phys Rev. D **10**, 428 (1974); Phys. Rev. D **10** 437 (1974).

[24] Y. Choquet-Bruhat and J.W. York, "The Cauchy Problem," in *General Relativity and Gravitation, I*, ed. A. Held (Plenum: New York, 1980), 99.

[25] C. Teitelboim, Ann. Phys. (NY) **79**, 542 (1973).

[26] C. Teitelboim, Ph.D. thesis, Princeton University, 1973 (unpublished); Phys. Rev. Lett. **38**, 1106 (1977).

[27] A. Anderson and J.W. York, in preparation.